# ON INITIAL VALUE PROBLEM FOR IMPLICIT DYNAMIC EQUATIONS ON TIME SCALES

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***Summary:*** *This paper is concerned with the index concept; the unique solvability of the initial value problem for implicit dynamic equation on time scale. This result is a generalization of previous results for differential and difference-algebraic equations.*

*Key words and phrases. Time scales, Implicit dynamic equation, Index of a system, Boundary-value problem, Green's function.*

## I. INTRODUCTION AND PRELIMINARIES

In last decades, the theory of linear differential-algebraic equations (LDAEs for short) has been an intensively discussed field in both theory and practice. The general form of LDAEs is

$$A_t\, x'(t) = \overline{B_t}\, x(t) \tag{1.1}$$

where $A.$ and $\overline{B}.$ are given matrix functions. The equation (1.1) can be seen in many real problems, such as in electric circuits, chemical reactions, vehicle systems...

Together with the theory of differential-algebraic equations, there has been a great interest in singular difference equation (SDE)

$$A_n\, x(n+1) = \overline{B}_n\, x(n) \tag{1.2}$$

which was introduced in [6, 10] and the solvability of IVPs as well as multipoint BVPs are studied in [1, 2]. This implicite difference equation appears in many practical areas, such as the Leontiev dynamic model of multisector economy, the Leslie population growth model, singular discrete optimal control problems and so forth (see [4, 5]).

Further, in recent years, to unify continuous and discrete analysis or to describe the processing of numerical calculation with non-constant steps, a new theory was born and is more and more extensively concerned, that is the theory of the analysis on time scales. The most popular examples of time scales are $T = \mathbb{R}$ and $T = \mathbb{Z}$. Using "language" of time scales, we rewrite the equations (1.1) and (1.2) under the form

$$A_t\, x^{\Delta} = \overline{B_t}\, x, \tag{1.3}$$

with $t$ in time scale $T$ and $\Delta$ is the differential operator on $T$.

Equation (1.3) still appears in the computational processes with non constant steps. Naturally, the question arises whether results for the equations (1.1) and (1.2) can be extended and unified for the implicit dynamic equations of the form (1.3). The purpose of this paper is to answer a part of this question, those are the solvability of the Cauchy problem and some matters concerning the boundary-value problem of the equation (1.3).

## II. PRELIMINARIES

This section surveys some notations on the theory of the analysis on time scales which was introduced by Stefan Hilger 1988 [9]. A time scale is a nonempty closed subset of the real numbers $\mathbb{R}$, and we usually denote it by the symbol $T$. We assume throughout that a time scale $T$ is endowed with the topology inherited from the real numbers with the standard topology. We define the forward jump operator and the backward jump operator $\sigma, \rho : T \to T$ by $\sigma(t) = \inf\{s \in T : s > t\}$ (supplemented by $\inf \phi = \sup T$) and $\rho(t) = \sup\{s \in T : s < t\}$ (supplemented by $\sup \phi = \inf T$). The graininess $\mu : T \to \mathbb{R}^{+} \bigcap \{0\}$ is given by $\mu(t) = \sigma(t) - t$. A point $t \in T$ is said to be right-dense if $\rho(t) = t$, right-scattered if $\sigma(t) > t$, left-dense if $\rho(t) = t$, left-scattered if $\rho(t) < t$, and isolated if $t$ is right-scattered and left-scattered. For every $a, b \in T$, by $[a, b]$, we mean the set $\{t \in T : a \le t \le b\}$. The set $T^k$ is defined to be $T$ if $T$ does not have a left-scattered maximum; otherwise it is $T$ without this left-scattered maximum. Let $f$ be a function defined on $T$, valued in $\mathbb{R}^m$. We say that $f$ is delta differentiable (or simply: differentiable) at $t \in T^k$ provided there exists a vector $f^{\Delta}(t) \in \mathbb{R}^m$, called the derivative of $f$, such that for all $\varepsilon > 0$ there is a neighborhood $V$ around $t$ with $\left\| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right\| \le \varepsilon |\sigma(t) - s|$, $\forall s \in V$ for all s ∈ V. If f is differentiable for every $t \in T^k$, then $f$ is said to be differentiable on $T$. If $T = \mathbb{R}$ then delta derivative is $f'(t)$ from continuous calculus; if $T = \mathbb{Z}$, the delta derivative is the forward difference, $\Delta f$, from discrete calculus. A function $f$ defined on $T$ is $rd-$continuous if it is continuous at every right-dense point and if the left-sided limit exists at every left-dense point. The set of all $rd-$continuous function from $T$ to a Banach space $X$ is denoted by $C_{rd}(T, X)$. A matrix function $f$ from $T$ to $\mathbb{R}^m \times \mathbb{R}^m$ is said to be regressive if $\det(I + \mu(t) f(t)) \neq 0$ for every $t \in T$.

Theorem 2.1 (see [3]). Let $t \in T$ and let $A_t$ be a $rd-$continuous $m \times m$ matrix function. Then, for any $t_0 \in T^k$, IVP

$$x^{\Delta} = A_t x, \ x(t_0) = x_0 \tag{2.4}$$

has a unique solution $x(.)$ defined on $t \ge t_0$. Further, if $A_t$ is regressive, this solution exists on $t \in T$.

The solution of the corresponding matrix-valued IVP $X^{\Delta} = A_t X, X(s) = I$ always exists for $t \ge s$, even $A_t$ is not regressive. In this case, $\Phi_A(t, s)$ is defined only with $t \ge s$ (see [8]) and is called the Cauchy operator of the dynamic equation (2.4). If we suppose further that $A_t$ is regressive, the Cauchy operator $\Phi_A(t, s)$ is defined for all $s, t \in T$.

It is seen that any solution $x(.)$ of the dynamic equation (2.4) can be written as $x(.) = \Phi_A(., t_0) x_0$ and the cocycle property: $\Phi_A(t, \tau) = \Phi_A(t, s) \Phi_A(s, \tau)$ is valid for all $\tau \le s \le t$.

Such as, if A is a constant then $\Phi_A(t, s) = \exp\{A \ \mathrm{mes}([t, s])\} \prod_{s \le \tau < t, \mu(\tau) \neq 0} (I + \mu(\tau) A)$, where

$\text{mess}(.)$ is Lebesgue measure on $\mathbb{R}$and $[a,b]$ is an interval on $T$.

## III. LINEAR IMPLICIT DYNAMIC EQUATIONS ON TIME SCALES

Let T be a time scale. We consider the homogeneous equation

$$A_t x^\Delta = \bar{B}_t x, \tag{3.1}$$

where $A, \bar{B} \in C_{rd}(T^k, \mathbb{R}^{m\times m})$. In the case where the matrices $A_t$ are invertible for every $t \in T$, we can multiply both sides of (3.1) by $A_t^{-1}$ to obtain an ordinary dynamic equation

$$x^\Delta = A_t^{-1}\bar{B}_t x, \quad t \in T$$

which has been well studied. If there is at least a $t$ such that $A_t$ is singular, we can not solve explicitly the leading term $x^\Delta$. In fact, we are concerned with a so-called ill-posed problem where the solutions of Cauchy problem may exist only on a submanifold or even they do not exist. One of the ways to solve this equation is to impose some further assumptions stated under the form of indices of the equation.

Denote $\alpha(t) = \begin{cases} t & ; \quad t \ \text{right}-\text{dense} \\ \rho(t) & ; \quad t \ \text{right}-\text{scattered} \end{cases}$

We introduce the so-called index-1 of the equation (3.1). Suppose that $\text{rank}A_t = r$ for all $t \in T$ and that $\ker A_t$ depends smoothly on $t$, i.e., there is a projector function $Q_t$ to be smooth and $\text{im}Q_t = \ker A_t$. Moreover, $Q_t^\Delta$ is $\text{rd}-\text{continuous}$ on $T^k$. Let $T_t \in GL(\mathbb{R}^m)$ such that $T_t|_{\ker A_t}$ is an isomorphism between $\ker A_t$ and $\ker A_{\alpha(t)}$; $T \in C_{rd}\left(T^k, \mathbb{R}^{m\times m}\right)$. This assumption will be satisfied if the matrix function $A_t$ are smooth. Indeed, let matrix $A_t$ possess a singular value decomposition

$$A_t = U_t \Sigma_t V_t^T,$$

where $U_t, V_t$ are orthogonal matrices and $\Sigma_t$ is a diagonal matrix with singular values $\sigma_t^1 \ge \sigma_t^2 \ge ... \ge \sigma_t^r > 0$ on its main diagonal. Since $A_t$ is smooth in $t$, on the above decomposition of $A_t$ we can choose the matrix $V_t$ to be smooth in $t$ (see [5]). Hence, by putting $Q_t = V_t \,\text{diag}(O, I_{m-r}) V_t^T$ and $T_t = V_{\alpha(t)} V_t^{-1}$, we obtain $Q_t$ and $V_t$ as the requirement.

Let $Q_t$ and $T_t$ be such matrices and put $P_t := I - Q_t$. From the relation $P_t x^\Delta(t) = (P_{\alpha(t)} x(t))^\Delta - (P_{\alpha(t)})^\Delta x(t)$ for all $t \in T^k$ we get

$$A_t x^\Delta(t) = A_t P_t x^\Delta(t) = A_t((P_{\alpha(t)} x(t))^\Delta - (P_{\alpha(t)})^\Delta x(t)).$$

Therefore, the implicit dynamic equation (3.1) can be rewritten as

$$\begin{bmatrix} A_t(P_{\alpha(t)}x)^\Delta = (A_t(P_{\alpha(t)}))^\Delta, \quad t \in T, \\ A_t(P_{\alpha(t)}x)^\Delta = B_t x, \quad t \in T \end{bmatrix} \tag{3.2}$$

where $B_t = A_t(P_{\alpha(t)})^{\Delta} + \overline{B}_t$. Thus, we should look for solutions of (3.1) from the space $C_N^1$:

$$C_N^1(T^k, \mathbb{R}^m) = \{x \in C_{rd}(T^k, \mathbb{R}^m): \ P_{\alpha(t)}x \ \text{ is differentiable}\}$$

Note that $C_N^1$ does not depend on the choice of the projector function since the relations $P_t P_t = P_t$ and $P_t P_t = P_t$ are true for each two projectors $P_t$ and $P_t$ along the space $\ker A_t$.

Let $S_t = \{x \in \mathbb{R}^m, \overline{B}_t x \in \mathrm{im}A_t\}$. The relation $\overline{B}_t x \in \mathrm{im}A_t \Leftrightarrow B_t x \in \mathrm{im}A_t$ implies

$$S_t = \{x \in \mathbb{R}^m, B_t x \in \mathrm{im}A_t\}.$$

Under these notations, we have:

Lemma 3.1. The following assertions are equivalent

i) kerAα(t) ∩ St = {0},

ii) The matrix $G_t = A_t - B_t T_t Q_t$ is nonsingular,

iii) $\mathbb{R}^m = \ker A_{\alpha(t)} \oplus S_t, \forall t \in T$.

Proof.

i) $\Rightarrow$ ii): Let $t \in T^k$ and $x \in \mathbb{R}^m$ such that $(A_t - B_t T_t Q_t)x = 0 \Leftrightarrow B_t(T_t Q_t x) = Ax$. This equation implies $T_t Q_t x \in S_t$. Since $\ker A_{\alpha(t)} \cap S_t = \{0\}$ and $T_t Q_t x \in \ker A_{\alpha(t)}$, it follows that $T_t Q_t x = 0$. Hence, $Q_t x = 0$ which implies $A_t x = 0$. This means that $x \in \ker A_t$. Thus, $x = Q_t x = 0$, i.e., the matrix $G_t = A_t - B_t T_t Q_t$ is nonsingular.

ii) $\Rightarrow$ iii): It is easy to see that $x = (I + T_t Q_t G_t^{-1} B_t)x - T_t Q_t G_t^{-1} B_t x$. Can see that $T_t Q_t G_t^{-1} B_t x \in \ker A_{\alpha(t)}$ and $B_t(I + T_t Q_t G_t^{-1} B_t)x = B_t x - (A_t - B_t T_t Q_t) G_t^{-1} B_t x + A_t G_t^{-1} B_t x$ $= A_t G_t^{-1} B_t x \in \mathrm{im}A_t$. Thus $(I + T_t Q_t G_t^{-1} B_t)x \in S_t$, and we have $\mathbb{R}^m = S_t + \ker A_{\alpha(t)}$.

Let $x \in \ker A_{\alpha(t)} \cap S_t$. Since $x \in S_t$, there is a $z \in \mathbb{R}^m$ such that $B_t x = A_t z = A_t P_t z$ and since $x \in \ker A_{\alpha(t)}$, $T_t^{-1} \in \ker A_t$. Therefore, $T_t^{-1} = Q_t T_t^{-1} x$. Hence, $(A_t - B_t T_t Q_t) T_t^{-1} x = -(A_t - B_t T_t Q_t) P_t z$ which follows that $T_t^{-1} x = -P_t z$. Thus, $T_t^{-1} x = 0$ and $x = 0$. So, we have iii).

iii) $\Rightarrow$ i) is obvious. Lemma 3.1 is proved. □

Lemma 3.2. Suppose that the matrix $G_t$ is nonsingular. Then, there hold the following assertions:

1. $P_t = G_t^{-1} A_t$,, (3.3)

2. $Q_t = -G_t^{-1} B_t T_t Q_t$, (3.4)

3. $Q_t := -T_tQ_tG_t^{-1}B_t$ is the projector onto $\ker A_{\alpha(t)}$ along $S_t$ (3.5)

4. a. $P_tG_t^{-1}B_t = P_tG_t^{-1}B_tP_{\alpha(t)},$ , (3.6)

b. $Q_tG_t^{-1}B_t = Q_tG_t^{-1}B_tP_{\alpha(t)} - T_t^{-1}Q_{\alpha(t)},$ (3.7)

5. $T_tQ_tG_t^{-1}$ does not depend on the choice of $T_t$ and $Q_t$ . (3.8)

Proof.

1. Noting that $G_tP_t = (A_t - B_tT_tQ_t)P_t = A_tP_t = A_t$ we get (3.3).

2. From $B_tT_tQ_t = A_t - G_t$, it follows $G_t^{-1}B_tT_tQ_t = P_t - I = -Q_t$. Thus, we have (3.4).

3. $Q_t^2 = T_tQ_tG_t^{-1}B_tT_tQ_tG_t^{-1}B_t \overset{(3.4)}{=} -T_tQ_tQ_tG_t^{-1}B_t = -T_tQ_tG_t^{-1}B_t = Q_t$ and $A_{\alpha(t)}Q_t = -A_{\alpha(t)}T_tQ_tG_t^{-1}B_t = 0$. This means that $Q_t$ is a projector onto $\ker A_{\alpha(t)}$. From the proof of iii), Lemma 3.1, we see that $Q_t$ is the projector onto $\ker A_{\alpha(t)}$ along $S_t$.

4. Since $T_t^{-1}Q_{\alpha(t)}x \in \ker A_t$ for any $x$

$$P_tG_t^{-1}B_tQ_{\alpha(t)} = P_tG_t^{-1}B_tT_tT_t^{-1}Q_{\alpha(t)} = -P_tG_t^{-1}(A_t - B_tT_tQ_t)Q_tT_t^{-1}Q_{\alpha(t)} = 0$$

Therefore, $P_tG_t^{-1}B_t = P_tG_t^{-1}B_tP_{\alpha(t)}$ so we have (3.6). Finally,

$$Q_tG_t^{-1}B_t = Q_tG_t^{-1}B_tP_{\alpha(t)} + Q_tG_t^{-1}B_tT_tQ_tT_t^{-1}Q_{\alpha(t)} = Q_tG_t^{-1}B_tP_{\alpha(t)} -$$

$$-Q_tG_t^{-1}(A_t - B_tT_tQ_t)Q_tT_t^{-1}Q_{\alpha(t)} = Q_tG_t^{-1}B_tP_{\alpha(t)} - Q_tT_t^{-1}Q_{\alpha(t)} = Q_tG_t^{-1}B_tP_{\alpha(t)} - T_t^{-1}Q_{\alpha(t)}$$

Thus, we get (3.7).

5. Let $T_t'$ be an other linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^m$ satisfying $T_t'|_{\ker A_t}$ to be an isomorphism from $\ker A_t$ to $\ker A_{\alpha(t)}$ and $Q'_t$ be a projection onto $\ker A_t$. Denote $B'_t = A_t(P'_{\alpha(t)})^{\Delta} + \bar{B}_t$ and $G'_t = A_t - B'_tT'_tQ'_t$. It is easy to see that

$$T_tQ_tG_t^{-1}G'_t = T_tQ_tG_t^{-1}(A_t - (A_t((P'_{\alpha(t)})^{\Delta} - (P_{\alpha(t)})^{\Delta}) + B_t)T'_tQ'_t) =$$

$$T_tQ_tP_t - T_tQ_tP_t((P'_{\alpha(t)})^{\Delta} - (P_{\alpha(t)})^{\Delta})T'_tQ'_t - T_tQ_tG_t^{-1}B_tT'_tQ'_t = -T_tQ_tG_t^{-1}B_tT'_tQ'_t \overset{(3.5)}{=} T'_tQ'_t$$

Therefore, $T_tQ_tG_t^{-1} = T'_tQ'_tG_t'^{-1}$. Lemma 3.2 is proved □

Definition 3.3. The LIDE (3.1) is said to be index-1 if for all $t \in T$, the following conditions hold:

i) $\operatorname{rank} A_t = r = \text{constant} \quad (1 \le r \le m-1),$ ),

ii) $\ker A_{\alpha(t)} \cap S_t = \{0\}$.

Throughout of this paper, suppose that the LIDE (3.1) is of index-1. We are in position to deal with the way to solve the equation (3.1).

Theorem 3.4. We can split the equation (3.1) into a dynamic equation on time scale and an algebraic equation.

Proof.

Multiplying (3.2) by $P_tG_t^{-1}$ and $Q_tG_t^{-1}$, respectively, it yields

$$\begin{cases} P_tG_t^{-1}A_t(P_{\alpha(t)}x)^{\Delta} = P_tG_t^{-1}B_tx \\ Q_tG_t^{-1}A_t(P_{\alpha(t)}x)^{\Delta} = Q_tG_t^{-1}B_tx. \end{cases}$$

Applying (3.3) of Lemma 3.2, we have

$$\begin{cases} P_t(P_{\alpha(t)}x)^{\Delta} = P_tG_t^{-1}B_tx \\ \quad 0 \quad\quad = Q_tG_{tB_tx.}^{-1} \end{cases}$$

(3.9)

From (3.6) and (3.7) of Lemma 3.2, the equation (3.9) becomes

$$\begin{cases} (P_{\alpha(t)}x)^{\Delta} = ((P_{\alpha(t)})^{\Delta} + P_tG_t^{-1}B_t)(P_{\alpha(t)}x) \\ Q_{\alpha(t)}x = T_tQ_tG_t^{-1}B_tP_{\alpha(t)}x \end{cases} \tag{3.10}$$

Therefore, by denoting $u(t) = P_{\alpha(t)}x(t),\ v(t) = Q_{\alpha(t)}x(t)$, the equation (3.10) can be rewritten

$$\begin{cases} u^{\Delta} = ((P_{\alpha(t)})^{\Delta} + P_tG_t^{-1}B_t)u \\ v = T_tQ_tG_t^{-1}B_tu \end{cases} \tag{3.11}$$

Let $t_0 \in T^k$. Solving $u(t)$ from the first equation of (3.11) with the initial condition $u(t_0) = P_{\alpha(t_0)}x_0$ and using the second relation of (3.11), we get an expression of the solutions of the index-1 LIDE (3.1) $x(t) = u(t) + v(t)$ for $t \geq t_0$.

Inspired by the above decoupling procedure, we state initial conditions for the index-1 LIDE (3.1) as

$$P_{\alpha(t_0)}(x(t_0) - x_0) = 0,\ x_0 \in \mathbb{R}^m \text{ given} \tag{3.12}$$

It yields that $u(t_0) = P_{\alpha(t_0)}x(t_0) = P_{\alpha(t_0)}x_0$, but we do not expect $x(t_0) = x_0$ as in the case of ordinary dynamic equations on time scales. Theorem 3.4 is proved. □

Remark 3.5.

i) Denoting $Q_{tcan} := -T_tQ_tG_t^{-1}B_t,\ P_{tcan} := I - Q_{tcan}$. From (3.5) of Lemma 3.2, $Q_{tcan}$ projects onto $\ker A_{\alpha(t)}$ along $S_t$ and is called the canonical projector for the index-1 case. Note that $Q_{tcan}$ is $rd-$ continuous and independent from the choice of $Q_t$ and $T_t$. The solutions of (3.1) with the initial condition (3.12) are represented by

$$x(t) = P_{\alpha(t)}x(t) + Q_{\alpha(t)}x(t) = (I + T_tQ_tG_t^{-1}B_t)u = P_{tcan}u(t), \quad t \ge t_0, \tag{3.13}$$

where $u \in C^1_{rd}$ solves from the inherent ordinary dynamic equation (3.11) with the initial condition $u(t_0) = P_{\alpha(t_0)}x_0$. From (3.8) of Lemma 3.2, this expression of $x$ does not depend on the choice of $T_t$ and $Q_t$.

By multiplying both sides of the homogeneous equation associated to (3.11) with $Q_t$ and using the fact: $0 = (Q_{\alpha(t)}P_{\alpha(t)})^\Delta = Q_t(P_{\alpha(t)})^\Delta + (Q_{\alpha(t)})^\Delta P_{\alpha(t)} \Rightarrow Q_t(P_{\alpha(t)})^\Delta = -(Q_{\alpha(t)})^\Delta P_{\alpha(t)}$,

it yields $Q_tu^\Delta = Q_t(P_{\alpha(t)})^\Delta P_{tcan}u = -(Q_{\alpha(t)})^\Delta P_{\alpha(t)}P_{tcan}u = -(Q_{\alpha(t)})^\Delta P_{\alpha(t)}u.$

Further, from $Q_tu^\Delta = (Q_{\alpha(t)}u)^\Delta - (Q_{\alpha(t)})^\Delta u$ we get $(Q_{\alpha(t)}u)^\Delta = (Q_{\alpha(t)})^\Delta (Q_{\alpha(t)}u).$

Hence, if $Q_{\alpha(t_0)}u(t_0) = 0$ then $Q_{\alpha(t)}u(t) = 0$ for all $t \ge t_0$. Therefore, the equation (3.11) has the invariant property: if $x(t_0) \in \mathrm{im}P_{\alpha(t_0)}$ then $x(t) \in \mathrm{im}P_{\alpha(t)}$ for all $t \in T^k$.

ii) When $T = \mathbb{R}$, $(\alpha(t) = t, \ \forall t \in \mathbb{R})$ we choose $T_t = -i_d$ to see the result mentioned in [7]. For the case $T = \mathbb{Z}$, the result can be seen in [1].

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