A MATHEMATICAL MODEL FOR RECTIFIER CIRCUITS USING SEMICONDUCTOR DIODES

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Abstract: *In previous studies, mathematicians have shown that, the rectifier circuit uses semiconductor diode, has been simulated by discontinuous differential equations. However, because of this discontinuity, the equation cannot be solved, even by numerical methods.*

The mathematical model for rectifier is set up in this paper to replace the discontinuous differential equation, which is mentioned above. The properties of the rectifier circuit using semiconductor diodes presented by the differential inclusions are considered by analyzing the mathematical model received. This is significant in the mathematical point of view, because describing and studying the stability of solutions of differential inclusions is much easier and more explicit than the discontinuous differential equations.

Based on the results of this study, we hope to get more profound results in further studies and investigate an optimal process for an assembly line of rectifiers in electrical engineering.

Keywords. *rectifier circuit, differential inclusions, semiconductor diode, mathematical model.*

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I. INTRODUCTION

The emergence of mathematical models has addressed a large number of applied problems, such as mechanics, electricity, theory of automation and control, struggle for survival in ecological systems, ... Mathematics is the tool for describing changes in each domain as dynamic systems, through which one can indicate their characteristics. Currently, the research in this area is still very developed. One of the problems that attracts attention is to study by mathematical modelling an operation of rectifier circuits (see [1] - [7]).

As we already know, most electrical installations use direct current, but the power source is alternating current. Therefore, rectifiers are very important, indispensable and widely used in the electrical industry. A rectifier is an electric circuit consisting of electrical components used to convert alternating current to direct current. This research has led to many interested results (see [8], [9]).

In this paper we will research a mathematical model for rectifier circuits using diodes. The rectifier circuit has the following general form:

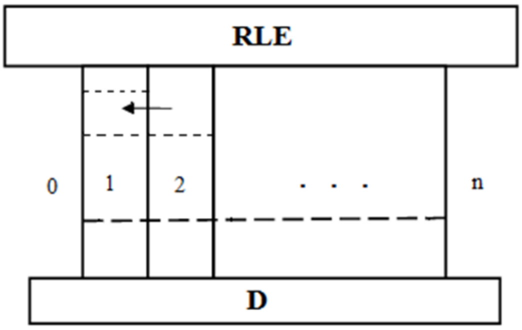

Figure 1. *The RLDE circuit*

The model describes the operation of the circuit will be presented by differential inclusions that is defined as

$$\begin{cases} \dfrac{dX}{dt} \in F(t) - \Im X - N_K X, \\ X \in K, \end{cases} \tag{1}$$

where, a set K is a cone in the space $\mathbb{R}^n$; $X(t)$ is an unknown function whose values belong to $\mathbb{R}^n$ at moment t; $\Im$ is a known constant square matrix of order n; $F(t)$ is a known continuous vector function with its values in $\mathbb{R}^n$ and the set $N_K X$ is called the normal cone which is defined by

$$N_K X = \{Z \in \mathbb{R}^n : (Z,\ \xi - X) \le 0,\ \forall \xi \in K\}. \tag{2}$$

We know that, the theory of differential inclusions and their applications is an intensively developed field of mathematics since the mid-19th century to now. There have been many studies showing that differential inclusions are equivalent to some differential equations with discontinuous right hand sides, such as in [1]. These studies help to find solutions of differential inclusions. At that, the solution of the system (1) is understood as a locally absolutely function which satisfies (1) almost everywhere.

The main content of this paper is showed in a theorem that gives a mathematical model for rectifier circuits. At that, the model is presented by differential inclusions of the form (1).

II. THE MATHEMATICAL MODEL FOR RECTIFIER CIRCUITS

Based on circuit theory (see [6], [7]), as we know, branch of a circuit diagram is two terminals of an element; point of connection between two or more branches is called node. Moreover, if i, u are currents and voltages across branches of any selected tree Υ and I, U are currents and voltages across branches complementing the tree Υ to the original circuit diagram, then we have

$$\begin{cases} U = Mu \\ i = -M^T I, \end{cases} \tag{3}$$

where M^T is the transposition matrix of M.

Let us consider an electrical circuit having a circuit diagram S and including resistances, inductances and a diode converter D. At that, the diode converter D contains m diodes. In each diode, positive current readily goes from the anode to the cathode. We denote by $x_j,\ y_j\ (j = \overline{1,m})$, respectively, the current and the voltage across the $j-th$ diode. Assume that diodes are ideal, that is, their currents x_j and voltages y_j are satisfied by

$$\begin{cases} x_j \geq 0 \\ y_j \leq 0 \quad ; \ j = \overline{1,m}. \\ x_j\, y_j = 0 \end{cases} \tag{4}$$

Note $x = (x_1, x_2, ..., x_m)$ and $y = (y_1, y_2, ..., y_m)$ then from (4), it easily follows

$$x \in \mathbb{R}_+^m,\ y \in \mathbb{R}_-^m \text{ and } (x, y) = 0\,. \tag{5}$$

Now, we formulate and prove a theorem called the theorem on the mathematical model for rectifiers circuits.

Theorem. *The mathematical model for rectifier circuits is presented by differential inclusions of the form* (1) *in which a function* $F(t)$, *a matrix* $\Im$ *and a set* K *are defined in the proof process of the theorem.*

Proof: In the circuit diagram S all nodes are numbered in some order from 0 to n. We denote by $i_k,\ u_k,\ \left(k = \overline{0,n}\right)$ respectively, the current passing the $k-th$ node, the voltage between the node k and node 0. After that, we are interested in vectors $i_D = (i_1, i_2, ..., i_n)$ and $u_D = (u_1, u_2, ..., u_n)$ (the vectors i_0 and u_0 are not interested, because they are presented through, respectively, other currents, other voltages). In order to show this, let Υ_1 denote a tree consists of all nodes. By the first Kirchhoff's law, we have $\sum_{j=1}^{m} a_{kj} x_j = i_k,\ k = \overline{1,n}$; consequently,

$$Ax = i_D \tag{6}$$

where $A = (a_{kj})_{n\times m}$ is a matrix whose elements receive values $1, -1$ and 0, respectively, if $j-th$ diode's anode is connected with the $k-th$ node, $j-th$ diode's cathode is connected with the $k-th$ node and in other cases: $a_{kj}=\begin{cases}1\\-1\\0\end{cases};\ \left(k=\overline{1,\,n}\,;\ j=\overline{1,\,m}\right).$ (7)

We can see that A is a matrix of one linear operator $(.)\colon\ \mathbb{R}^m\to\ \mathbb{R}^n$, satisfies (6). We note by A^{-1} a matrix of an inverse operator $(.)^{-1}$, A^T is the transpose matrix of A.
On the other hand, using (3) we obtain

$$A^T u_D = y. \tag{8}$$

Now, we denote by Υ_2 a tree containing resistances R, inductances L and a supply source. Then, branches complementing the tree Υ_2 to the original circuit diagram $\mathbb{S}$ are included resistances r, inductances l and the diode converter D. By the second Kirchhoff's law, we have

$$U_R = M_1 u_r, \tag{9}$$

$$U_L = M_2 u_r + M_3 u_l + M_4 u_D + E(t), \tag{10}$$

where, $E(t)$ depends on the voltage $e(t)$ of the supply source; M_1, M_2, M_3, M_4 are matrices that depend on the research circuit; U_L, U_R are potential difference L, R, respectively; u_l, u_r are potential difference $l,\ r$, respectively.

From (9) and (10), it implies that

$$\begin{pmatrix}U_R\\U_L\end{pmatrix}=\begin{pmatrix}M_1 & 0 & 0\\ M_2 & M_3 & M_4\end{pmatrix}.\begin{pmatrix}u_r\\u_l\\u_D\end{pmatrix}+\begin{pmatrix}0\\E(t)\end{pmatrix}.$$

Using the last equation and (3), we get

$$\begin{pmatrix}i_r\\i_l\\i_D\end{pmatrix}=-\begin{pmatrix}M_1^T & M_2^T\\ 0 & M_3^T\\ 0 & M_4^T\end{pmatrix}.\begin{pmatrix}I_R\\I_L\end{pmatrix}.$$

Where $I_L,\ I_R;\ i_l,\ i_r$ respectively are current intensity through $L, R;\ l,\ r$.

From here, we get

$$i_D = -M_4^T I_L, \tag{11}$$

$$i_l = -M_3^T I_L, \tag{12}$$

$$i_r = -M_2^T I_L - M_1^T I_R. \tag{13}$$

In order to find the mathematical model for rectifier circuits, we use obvious following equations

$$U_R = RI_R, \tag{14}$$

$$u_r = ri_r, \tag{15}$$

$$L\frac{dI_L}{dt} = U_L, \tag{16}$$

$$l\frac{di_l}{dt} = u_l. \tag{17}$$

Where L, I, R and r are diagonal matrices whose diagonal elements are positive values. To solve the system (6) - (17), we consider I_L and u_D as the main unknowns. Further, by (12) and (17) we obtain

$$M_3 u_l = -M_3 l M_3^T \frac{dI_L}{dt}. \tag{18}$$

On the other hand, from equations (9), (13), (14) and (15), it implies

$$u_r = r\, i_r = -r\left(M_2^T I_L + M_1^T I_R\right) = -r\, M_2^T I_L - r\, M_1^T R^{-1} M_1 u_r.$$

Thus, with I is the identity matrix, we have

$$u_r = -\left(I + r\, M_1^T R^{-1} M_1\right)^{-1} r\, M_2^T I_L. \tag{19}$$

Using (10), (16), (18) and (19), characteristics of the research circuit are represented by

$$\Theta\frac{dI_L}{dt} + \mathrm{B}\, I_L - M_4 u_D = E(t) \tag{20}$$

here $\Theta := L + M_3 l M_3^T$ and $\mathrm{B} := M_2\left(I + r\, M_1^T R^{-1} M_1\right)^{-1} r\, M_2^T I_L$.

Note that

$$X = \Theta^{\frac{1}{2}} I_L;\ Y = \Theta^{-\frac{1}{2}}\left(-M_4\right) u_D. \tag{21}$$

then, the equation (20) is written by

$$\frac{dX}{dt} + \Im X + Y = F(t), \tag{22}$$

$$\begin{cases} \Im = \Theta^{-\frac{1}{2}} \mathrm{B} \Theta^{-\frac{1}{2}}, \\ F(t) = \Theta^{-\frac{1}{2}} E(t). \end{cases} \tag{23}$$

In order to finish the proof of the theorem we will study properties of X and Y.

First, by (21) we obtain

$$(X,Y) = \left(\Theta^{\frac{1}{2}} I_L,\ \Theta^{-\frac{1}{2}} (-M_4) u_D \right) = \left(I_L, \left(\Theta^{\frac{1}{2}} \right)^T \Theta^{-\frac{1}{2}} (-M_4) u_D \right).$$

Since the matrix Θ is diagonal, we can see: $(X,Y) = (I_L, -M_4 u_D)$.

Furthermore, using (11), (6) and (8), we have

$$(X,Y) = \left(-M_4^T I_L,\ u_D \right) = (i_D,\ u_D) = (Ax,\ u_D) = \left(i_D,\ A^T u_D \right) = (x,y).$$

And, by (5) we also obtain

$$(X,\ Y) = 0. \tag{24}$$

On the other hand, from (6), (8) and (5) it directly implies

$$i_D \in A\mathbb{R}_+^m. \tag{25}$$

Additionally, note that

$$K = \Theta^{\frac{1}{2}} \left(-M_4^T \right)^{-1} \left(A\mathbb{R}_+^m \right) \tag{26}$$

then, by using (25), (21) and (11) we have

$$X \in K. \tag{27}$$

Finally, we will prove that $Y \in N_K X$. For this, we estimate a value $(Y,\ \xi - X),\ \forall \xi \in K$. From (24), (26), (21) and (8) we get

$$(Y,\ \xi - X) = (Y,\ \xi) = \left(\Theta^{-\frac{1}{2}} (-M_4) u_D,\ \Theta^{\frac{1}{2}} \left(-M_4^T \right) Aa \right)$$

$$= \left(\left(\Theta^{\frac{1}{2}} \right)^T \Theta^{-\frac{1}{2}} (-M_4) u_D, \left(-M_4^T \right)^{-1} Aa \right) = \left((-M_4) u_D,\ \left(-M_4^T \right)^{-1} Aa \right)$$

$$= \left(u_D,\ \left(-M_4^T \right) \left(-M_4^T \right)^{-1} Aa \right) = (u_D,\ Aa) = \left(A^T u_D,\ a \right) = (y,a),$$

where $a \in \mathbb{R}_+^m$ such that $\Theta^{\frac{1}{2}}\left(-M_4^T\right)^{-1} Aa = \xi.$

Consequently, $\left(Y,\, \xi - X\right) = \left(y, a\right) \le 0,\ \forall \xi \in K$ (because $a \in \mathbb{R}_+^m$ and $y \in \mathbb{R}_-^m$). From here and the definition $N_K X$ by (2), it implies that

$$Y \in N_K X. \tag{28}$$

Thus, using (22), (27) and (28), we obtain

$$\frac{dX}{dt} \in F\left(t\right) - \Im X - N_K X.$$

where $F\left(t\right),\ \Im,\ K$ are defined by (23) and (26). That completes the proof of the theorem.

To illustrate the results of the study, we consider a electric circuit of a following figure known as a full wave rectifier; it contains 4 diodes, a source, resistance R and inductance L. In a supply circuit there is a source including a voltage $e\left(t\right)$, resistance r and inductance l. This case is used to show the mathematical model of the form (1) for the considered rectifier circuit. For this, the choice of positive voltage is marked by indicators and on the other all nodes are numbered by $0, 1, 2, 3$ as the figure 2.

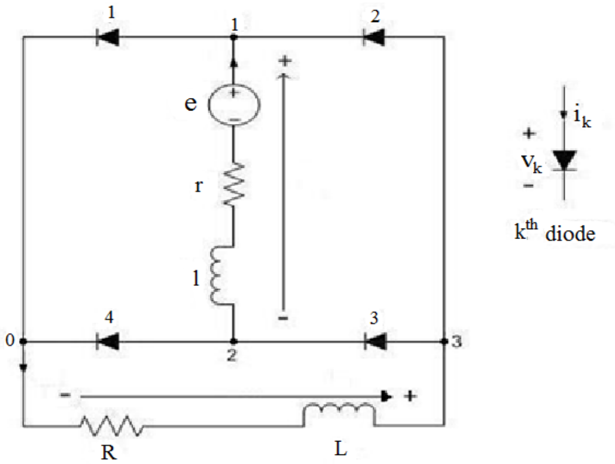

Figure 2. *The full wave rectifier*

We can see from (25) and (7) that

$$u_D = \left(u_1, u_2, u_3\right);\ \ i_D = \left(i_1, i_2, i_3\right) \in A\mathbb{R}_+^4$$

with the matrix A is determined by

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

One can easily see that a voltage of the supply circuit (between nodes 1 and 2) equals $u_1 - u_2$ and a voltage of the load circuit (between nodes 3 and 0) equals u_3. Then, by the Kirchhoff's laws we have $i_2 = i_3 - i_1$ and

$$\begin{cases} l\dfrac{di_1}{dt} + ri_1 + u_1 - u_2 = e(t) \\ L\dfrac{di_3}{dt} + Ri_3 + u_3 = 0. \end{cases} \tag{29}$$

From (21), with $\Theta = \begin{pmatrix} l & 0 \\ 0 & L \end{pmatrix}$; $-M_4 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, the system (29) is rewritten according to the equation of form (22), where $\Im = \begin{pmatrix} \dfrac{r}{l} & 0 \\ 0 & \dfrac{R}{L} \end{pmatrix}$ and $F(t) = \begin{pmatrix} \dfrac{e(t)}{l^{\frac{1}{2}}} \\ 0 \end{pmatrix}$.

Now, we have to find a set K such that $X \in K$ and $Y \in N_K X$.

First, we establish the set K is defined by (26): $\left(-M_4^T\right)\left(\Theta^{\frac{1}{2}}\right)^{-1} K = A\mathbb{R}_+^m$.

So, for every $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in K$, there exists $x = (x_i)_{4\times 1} \in \mathbb{R}_+^4$, $i = \overline{1,4}$ such that

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} \dfrac{1}{\sqrt{l}} & 0 \\ 0 & \dfrac{1}{\sqrt{l}} \end{pmatrix} . \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

So, in conjunction with (6) we deduce

$$\begin{pmatrix} \dfrac{1}{\sqrt{l}} X_1 \\ -\dfrac{1}{\sqrt{l}} X_1 \\ \dfrac{1}{\sqrt{L}} X_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ -x_3 + x_4 \\ x_2 + x_3 \end{pmatrix} = \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix}. \tag{30}$$

From here, we have

$$i_1 = x_1 - x_2,\ i_2 = -x_3 + x_4,\ i_3 = x_2 + x_3,\ i_1 = -i_2,$$

$$\begin{cases} i_3 \ge 0,\ i_3 + i_1 \ge 0,\ i_3 - i_1 \ge 0, \\ i_3 \ge \|i_1\|. \end{cases} \tag{31}$$

Then, with $X_1 = i_1\sqrt{l},\ X_2 = i_3\sqrt{L},$ we obtain

$$K = \left\{ X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \mathbb{R}^2 : X_2 \ge 0,\ X_2 \ge \sqrt{\frac{L}{l}} X_1,\ X_2 \ge -\sqrt{\frac{L}{l}} X_1 \right\}.$$

Finally, we have to show that $Y \in N_K X$. From (8), we have

$$\begin{cases} y_1 = \left(u_D, (1,0,0)\right) \le 0 \\ y_2 = \left(u_D, (-1,0,1)\right) \le 0 \\ y_3 = \left(u_D, (0,-1,1)\right) \le 0 \\ y_4 = \left(u_D, (0,1,0)\right) \le 0, \end{cases}$$

and we also have: $u_1 \le 0,\ u_3 \le u_1,\ u_3 \le u_2,\ u_2 \le 0,\ u_3 \le 0.$

From here it follows

$$\begin{cases} \frac{1}{\sqrt{L}} u_3 \le \frac{1}{\sqrt{L}} (u_1 - u_2), \\ \frac{1}{\sqrt{L}} u_3 \le \frac{1}{\sqrt{L}} (u_2 - u_1). \end{cases} \tag{32}$$

Moreover, using the note $Y = \Theta^{-\frac{1}{2}} (-M_4) u_D,$ we obtain

$$Y = \begin{pmatrix} \frac{u_1 - u_2}{\sqrt{l}} \\ \frac{u_3}{\sqrt{L}} \end{pmatrix} \in \mathbb{R}^2. \tag{33}$$

By (32) and (33), we have

$$Y \in N_K X = \left\{ \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \in \mathbb{R}^2 : Y_2 \le 0,\ Y_2 \le \sqrt{\frac{l}{L}} Y_1,\ Y_2 \le -\sqrt{\frac{l}{L}} Y_1 \right\},\ \forall X \in K.$$

Thus, the characteristics of the full wave rectifier are presented by differential inclusions of the form (1).

III. CONCLUSION

Mathematical simulation of engineering systems from which to study in an overview, the nature of their operating principle is one of the most important applications of mathematics. The characteristics of the rectifier using semiconductor diodes have been investigated in this paper by establishing a mathematical model that describes these characteristics and analyzes the mathematical models received. We have also considered a concrete case to illustrate the result of the study.

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